

WARPED PRODUCTS OF SINGULAR SEMIRIEMANNIAN MANIFOLDS

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ABSTRACT. This article studies the degenerate warped products of singular semi-Riemannian manifolds. One main result is that a degenerate warped product of semi-regular semi-Riemannian manifolds with the warping function satisfying a certain condition is a semi-regular semi-Riemannian manifold. The main invariants of the warped product are expressed in terms of those of the factor manifolds. Examples of singular semi-Riemannian manifolds which are semi-regular are constructed as warped products. Degenerate warped products are used to define spherical warped products. As applications, cosmological models and black holes solutions with semi-regular singularities are constructed. Such singularities are compatible with the densitized version of Einstein's equation, and don't block the time evolution. In following papers we will apply the technique developed here to resolve the singularities of the Friedmann-Lemaître-Robertson-Walker, Schwarzschild, Reissner-Nordström and Kerr-Newman spacetimes.

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Introduction

The warped product provides a way to construct new semi-Riemannian manifolds from known ones [3, 1, 9]. This construction has useful applications in General Relativity, in the study cosmological models and black holes. In such models, singularities are usually present, and at such points the warping function becomes 0. The metric of the product manifold in this case becomes degenerate, and we need to apply the tools of singular semi-Riemannian geometry.

This article continues the study of singular semi-Riemannian manifolds [7, 8], [12], extending it to warped products. After a brief recall of notions related to product manifolds in §1.1, basic notions of singular semi-Riemannian geometry and the main ideas from [12], which will be applied here, are remembered in §1.2. Then, in §2 we define the degenerate warped products of singular semi-Riemannian manifolds, and study the Koszul form of the warped product in terms of the Koszul form of the factors. The main results known from the literature about the non-degenerate warped products of semi-Riemannian manifolds are recalled in §3. Then, in §4 we show that the warped products of radical-stationary manifolds are also radical-stationary, if the warping function satisfies a certain condition. Then we prove a similar result for semi-regular manifolds, which ensures the smoothness of the Riemann curvature tensor. In §5 we express the Riemann curvature of semi-regular warped products in terms of the factor manifolds. Then, in §6, we introduce the polar and spherical warped products, which allows us to construct singular semi-Riemannian manifolds with radial or spherical symmetry.

We conclude in §7 by giving some examples of semi-regular warped products, and some applications to General Relativity. Cosmological models having the Big Bang singularity semi-regular, are proposed. Spherical solutions with semi-regular singularities are constructed in a general way. Semi-regular singularities are compatible with a densitized version of Einstein's equation, and they don't block the time evolution.

This article is part of a series developing the Singular Semi-Riemannian Geometry and its applications to the study of the singularities occurring in General Relativity.

1. Preliminaries

1.1. Product manifolds

We first recall some elementary notions about the *product manifold* $B \times F$ of two differentiable manifolds B and F (*cf. e.g.* [9], p. 24–25).

At each point $p = (p_1, p_2)$ of the manifold $M_1 \times M_2$ the tangent space decomposes as

$$(1) \quad T_{(p_1, p_2)}(M_1 \times M_2) \cong T_{(p_1, p_2)}(M_1) \oplus T_{(p_1, p_2)}(M_2),$$

where $T_{(p_1, p_2)}(M_1) := T_{(p_1, p_2)}(M_1 \times p_2)$ and $T_{(p_1, p_2)}(M_2) := T_{(p_1, p_2)}(p_1 \times M_2)$.

Let $\pi_i : M_1 \times M_2 \rightarrow M_i$, for $i \in \{1, 2\}$, be the canonical projections. The *lift of the scalar field* $f_i \in \mathcal{F}(M_i)$ is the scalar field $\tilde{f}_i := f_i \circ \pi_i \in \mathfrak{X}(M_1 \times M_2)$. The *lift of the vector field* $X_i \in \mathfrak{X}(M_i)$ is the unique vector field \tilde{X}_i on $M_1 \times M_2$ satisfying $d\pi_i(\tilde{X}_i) = X_i$. We denote the set of all vector fields $X \in \mathfrak{X}(M_1 \times M_2)$ which are lifts of vector fields $X_i \in \mathfrak{X}(M_i)$ by $\mathfrak{L}(M, M_i)$. The *lift of a covariant tensor* $T \in \mathcal{T}_s^0 M_i$ is given by $\tilde{T} \in \mathcal{T}_s^0(M_1 \times M_2)$, $\tilde{T} := \pi_i^*(T)$. The *lift of a tensor* $T \in \mathcal{T}_s^1 M_i$ is given, for any $X_1, \dots, X_s \in \mathfrak{X}(M_1 \times M_2)$, by $\tilde{T} \in \mathcal{T}_s^1(M_1 \times M_2)$, $\tilde{T}(X_1, \dots, X_s) = \tilde{X}$, where $\tilde{X} \in \mathfrak{X}(M_1 \times M_2)$ is the lifting of the vector field $X \in \mathfrak{X}(M_i)$, $X = T(\pi_i(X_1), \dots, \pi_i(X_s))$.

1.2. Singular semi-Riemannian manifolds

We recall here some notions about singular semi-Riemannian manifolds, and some of the main results from [12], which will be used in the rest of the article.

Definition 1.1. (also see [7]) A *singular semi-Riemannian manifold* (M, g) is a differentiable manifold M endowed with a symmetric bilinear

form $g \in \Gamma(T^*M \odot_M T^*M)$ named *metric*. The manifold (M, g) is said to be with *constant signature* if the signature of g is fixed, otherwise, (M, g) is said to be with *variable signature*. Particular cases are the *semi-Riemannian manifolds*, when the metric is non-degenerate, and *Riemannian manifolds*, when g is positive definite.

Definition 1.2. (also see [2], p. 1, [8], p. 3 and [9], p. 53) If (V, g) is a finite dimensional inner product space with an inner product g which may be degenerate, then we call the totally degenerate space $V_\circ := V^\perp$ the *radical* of V . The inner product g on V is non-degenerate if and only if $V_\circ = \{0\}$.

Definition 1.3. (see [7], p. 261, [10], p. 263) The *radical of TM* , denoted by $T_\circ M$, is defined by $T_\circ M = \cup_{p \in M} (T_p M)_\circ$. We denote by $\mathfrak{X}_\circ(M)$ the set of vector fields on M for which $W_p \in (T_p M)_\circ$. They form a vector space over \mathbb{R} and a module over $\mathcal{F}(M)$.

The remaining of this section recalls very briefly the main notions and results on singular semi-Riemannian manifolds, as presented in [12].

Definition 1.4. We define

$$(2) \quad T^\bullet M = \bigcup_{p \in M} (T_p M)^\bullet$$

where $(T_p M)^\bullet \subseteq T_p^* M$ is the space of covectors at p of the form $\omega_p(X_p) = \langle Y_p, X_p \rangle$ for some vectors $Y_p \in T_p M$ and any $X_p \in T_p M$. We define sections of $T^\bullet M$ by

$$(3) \quad \mathcal{A}^\bullet(M) := \{\omega \in \mathcal{A}^1(M) | \omega_p \in (T_p M)^\bullet \text{ for any } p \in M\}.$$

Definition 1.5. On $T^\bullet M$ there is a unique non-degenerate inner product g_\bullet , defined by $\langle\langle \omega, \tau \rangle\rangle_\bullet := g_\bullet(\omega, \tau) := \langle X, Y \rangle$, where $X^\bullet = \omega$, $Y^\bullet = \tau$, $X, Y \in \mathfrak{X}(M)$.

Definition 1.6. A tensor T of type (r, s) is named *radical-annihilator* in the l -th covariant slot if $T \in \mathcal{T}_{l-1}^r M \otimes_M T^\bullet M \otimes_M \mathcal{T}_{s-l}^0 M$.

Definition 1.7. We now show how to define uniquely the *covariant contraction* or *covariant trace*. We define it first on tensors $T \in T^\bullet M \otimes_M T^\bullet M$, by $C_{12}T = g_\bullet^{ab}T_{ab}$. This definition does not depend on the basis, because $g_\bullet \in T^{\bullet*} M \otimes_M T^{\bullet*} M$. This operation can be extended by linearity to any tensors which are radical in two covariant indices. For a tensor field T we define the contraction $C_{kl}T$ by

$$T(\omega_1, \dots, \omega_r, v_1, \dots, \bullet, \dots, \bullet, \dots, v_s).$$

If the metric is non-degenerate, we can define the covariant derivative of a vector field Y in the direction of a vector field X , where $X, Y \in \mathfrak{X}(M)$, by the *Koszul formula* (see *e.g.* [9], p. 61). If the metric is degenerate, we cannot extract the covariant derivative from the Koszul formula. We define the *Koszul form* as a shorthand for the long right part of the Koszul formula and were emphasized some of its properties.

Let's recall the definition of the Koszul form and its properties, without proof, from [12].

Definition 1.8 (The Koszul form). *The Koszul form* is defined as

$$\begin{aligned} \mathcal{K} : \mathfrak{X}(M)^3 &\rightarrow \mathbb{R}, \\ (4) \quad \mathcal{K}(X, Y, Z) &:= \frac{1}{2} \{ X\langle Y, Z \rangle + Y\langle Z, X \rangle - Z\langle X, Y \rangle \\ &\quad - \langle X, [Y, Z] \rangle + \langle Y, [Z, X] \rangle + \langle Z, [X, Y] \rangle \}. \end{aligned}$$

Theorem 1.9. Properties of the Koszul form of a singular semi-Riemannian manifold (M, g) :

- (1) Additivity and \mathbb{R} -linearity in each of its arguments.
- (2) $\mathcal{F}(M)$ -linearity in the first argument:
 $\mathcal{K}(fX, Y, Z) = f\mathcal{K}(X, Y, Z)$.
- (3) The *Leibniz rule*:
 $\mathcal{K}(X, fY, Z) = f\mathcal{K}(X, Y, Z) + X(f)\langle Y, Z \rangle$.
- (4) $\mathcal{F}(M)$ -linearity in the third argument:
 $\mathcal{K}(X, Y, fZ) = f\mathcal{K}(X, Y, Z)$.
- (5) It is *metric*:
 $\mathcal{K}(X, Y, Z) + \mathcal{K}(X, Z, Y) = X\langle Y, Z \rangle$.
- (6) It is *symmetric*:
 $\mathcal{K}(X, Y, Z) - \mathcal{K}(Y, X, Z) = \langle [X, Y], Z \rangle$.
- (7) Relation with the Lie derivative of g :
 $\mathcal{K}(X, Y, Z) + \mathcal{K}(Z, Y, X) = (\mathcal{L}_Y g)(Z, X)$.
- (8) $\mathcal{K}(X, Y, Z) + \mathcal{K}(Y, Z, X) = Y\langle Z, X \rangle + \langle [X, Y], Z \rangle$.

for any $X, Y, Z \in \mathfrak{X}(M)$ and $f \in \mathcal{F}(M)$. □

Definition 1.10. Let $X, Y \in \mathfrak{X}(M)$. The *lower covariant derivative* of Y in the direction of X is defined as the differential 1-form $\nabla_X^b Y \in \mathcal{A}^1(M)$

$$(5) \quad (\nabla_X^b Y)(Z) := \mathcal{K}(X, Y, Z)$$

for any $Z \in \mathfrak{X}(M)$. We also define the *lower covariant derivative operator*

$$(6) \quad \nabla^b : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathcal{A}^1(M)$$

which associates to each $X, Y \in \mathfrak{X}(M)$ the differential 1-form $\nabla_X^b Y$.

Definition 1.11 (see [8] Definition 3.1.3). A singular semi-Riemannian manifold (M, g) is *radical-stationary* if it satisfies the condition

$$(7) \quad \mathcal{K}(X, Y, \cdot) \in \mathcal{A}^\bullet(M),$$

for any $X, Y \in \mathfrak{X}(M)$.

Definition 1.12. Let $X \in \mathfrak{X}(M)$, $\omega \in \mathcal{A}^\bullet(M)$, where (M, g) is radical-stationary. The covariant derivative of ω in the direction of X is defined as

$$(8) \quad \nabla : \mathfrak{X}(M) \times \mathcal{A}^\bullet(M) \rightarrow A_d^1(M)$$

$$(9) \quad (\nabla_X \omega)(Y) := X(\omega(Y)) - \langle \nabla_X^b Y, \omega \rangle_\bullet,$$

where $A_d^1(M)$ denotes the set of 1-forms which are smooth on the regions of constant signature.

Definition 1.13. If the semi-Riemannian manifold (M, g) is radical-stationary, we define:

$$(10) \quad \mathcal{A}^{\bullet 1}(M) = \{\omega \in \mathcal{A}^\bullet(M) | (\forall X \in \mathfrak{X}(M)) \nabla_X \omega \in \mathcal{A}^\bullet(M)\},$$

$$(11) \quad \mathcal{A}^{\bullet k}(M) := \bigwedge_M^k \mathcal{A}^{\bullet 1}(M).$$

Definition 1.14. The *Riemann curvature tensor* is defined as

$$(12) \quad R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathbb{R},$$

$$(13) \quad R(X, Y, Z, T) := (\nabla_X \nabla_Y^b Z - \nabla_Y \nabla_X^b Z - \nabla_{[X, Y]}^b Z)(T)$$

for any vector fields $X, Y, Z, T \in \mathfrak{X}(M)$.

Definition 1.15. A singular semi-Riemannian manifold (M, g) satisfying

$$(14) \quad \nabla_X^b Y \in \mathcal{A}^{\bullet 1}(M)$$

for any vector fields $X, Y \in \mathfrak{X}(M)$ is called *semi-regular semi-Riemannian manifold*.

Proposition 1.16. A radical-stationary semi-Riemannian manifold (M, g) is semi-regular if and only if for any $X, Y, Z, T \in \mathfrak{X}(M)$

$$(15) \quad \mathcal{K}(X, Y, \bullet) \mathcal{K}(Z, T, \bullet) \in \mathcal{F}(M).$$

□

Example 1.17. We will construct a useful example of semi-regular metric [12]. Let's consider that there is a coordinate chart in which the metric is diagonal. The components of the Koszul form are in this case the Christoffel's symbols of the first kind, which are of the form $\pm \frac{1}{2} \partial_a g_{bb}$, because the metric is diagonal. Assume that $g = \sum_a \varepsilon_a \alpha_a^2 dx^a \otimes dx^a$, $\varepsilon_a \in \{-1, 1\}$. Then the metric is semi-regular if there is a smooth function $f_{abc} \in \mathcal{F}(M)$ with $\text{supp}(f_{abc}) \subseteq \text{supp}(\alpha_c)$ for any $a, b \in \{1, \dots, n\}$ and $c \in \{a, b\}$, and

$$(16) \quad \partial_a \alpha_b^2 = f_{abc} \alpha_c.$$

If $c = b$, $\partial_a \alpha_b^2 = 2\alpha_b \partial_a \alpha_b$ implies that the function is $f_{abb} = 2\partial_a \alpha_b$. In addition, this has to satisfy the condition $\partial_a \alpha_b = 0$ whenever $\alpha_b = 0$. We require the condition $\text{supp}(f_{abc}) \subseteq \text{supp}(\alpha_c)$ because for being semi-regular, a manifold has to be radical-stationary.

Theorem 1.18. The Riemann curvature of a semi-regular semi-Riemannian manifold (M, g) is a smooth tensor field $R \in \mathcal{T}_4^0 M$. \square

Proposition 1.19. The Riemann curvature of a semi-regular semi-Riemannian manifold (M, g) satisfies

$$(17) \quad \begin{aligned} R(X, Y, Z, T) &= X((\nabla_Y^b Z)(T)) - Y((\nabla_X^b Z)(T)) - (\nabla_{[X, Y]}^b Z)(T) \\ &\quad + \langle \nabla_X^b Z, \nabla_Y^b T \rangle \bullet - \langle \nabla_Y^b Z, \nabla_X^b T \rangle \bullet \end{aligned}$$

and

$$(18) \quad \begin{aligned} R(X, Y, Z, T) &= XK(Y, Z, T) - YK(X, Z, T) - K([X, Y], Z, T) \\ &\quad + K(X, Z, \bullet)K(Y, T, \bullet) - K(Y, Z, \bullet)K(X, T, \bullet) \end{aligned}$$

for any vector fields $X, Y, Z, T \in \mathfrak{X}(M)$. \square

2. Degenerate warped products of singular semi-Riemannian manifolds

The warped product is defined in general between two (non-degenerate) semi-Riemannian manifolds, (cf. [3], [1], [9], p. 204–211. It is straightforward to extend the definition to singular semi-Riemannian manifolds, as it is done in this section.

Definition 2.1 (generalizing [9], p. 204). Let (B, g_B) and (F, g_F) be two singular semi-Riemannian manifolds, and $f \in \mathcal{F}(B)$ a smooth function. The *warped product* of B and F with *warping function* f is the semi-Riemannian manifold

$$(19) \quad B \times_f F := (B \times F, \pi_B^*(g_B) + (f \circ \pi_B)\pi_F^*(g_F)),$$

where $\pi_B : B \times F \rightarrow B$ and $\pi_F : B \times F \rightarrow F$ are the canonical projections. It is customary to call B the *base* and F the *fiber* of the warped product $B \times_f F$.

We will use for all vector fields $X_B, Y_B \in \mathfrak{X}(B)$ and $X_F, Y_F \in \mathfrak{X}(F)$ the notation $\langle X_B, Y_B \rangle_B := g_B(X_B, Y_B)$ and $\langle X_F, Y_F \rangle_F := g_F(X_F, Y_F)$. The inner product on $B \times_f F$ takes, for any point $p \in B \times F$ and for any pair of tangent vectors $x, y \in T_p(B \times F)$, the explicit form

$$(20) \quad \langle x, y \rangle = \langle d\pi_B(x), d\pi_B(y) \rangle_B + f^2(p) \langle d\pi_F(x), d\pi_F(y) \rangle_F.$$

Remark 2.2. Definition 2.1 is a generalization of the warped product definition, which is usually given for the case when both g_B and g_F are non-degenerate and $f > 0$ (see [3], [1] and [9]). In our definition these restrictions are dropped.

Remark 2.3 (similar to [9], p. 204–205). For any $p_B \in B$, $\pi_B^{-1}(p_B) = p_B \times F$ is named the *fiber* through p_B and it is a semi-Riemannian manifold. $\pi_F|_{p_B \times F}$ is a (possibly degenerate) homothety onto F . For each $p_F \in F$, $\pi_F^{-1}(p_F) = B \times p_F$ is a semi-Riemannian manifold named the *leave* through p_F . $\pi_B|_{B \times p_F}$ is an isometry onto B . For each $(p_B, p_F) \in B \times F$, $B \times p_F$ and $p_B \times F$ are orthogonal at (p_B, p_F) . For simplicity, if a vector field is a lift, we will use sometimes the same notation if they can be distinguished from the context. For example, we will be using $\langle V, W \rangle_F := \langle \pi_F(V), \pi_F(W) \rangle_F$ for $V, W \in \mathfrak{L}(B \times F, F)$.

The following proposition recalls some evident facts used repeatedly in the proofs of the properties of warped products in [9], p. 24–25, 206.

Proposition 2.4. Let $B \times_f F$ be a warped product, and let be the vector fields $X, Y, Z \in \mathfrak{L}(B \times F, B)$ and $U, V, W \in \mathfrak{L}(B \times F, F)$. Then

- (1) $\langle X, V \rangle = 0$.
- (2) $[X, V] = 0$.
- (3) $V \langle X, Y \rangle = 0$.
- (4) $X \langle V, W \rangle = 2f \langle V, W \rangle_F X(f)$.

Proof. (1) and (2) are evident because the manifold is $B \times F$.

(3) $\langle X, Y \rangle = \langle X, Y \rangle_B$ is constant on fibers, and $V \langle X, Y \rangle = 0$ because V is vertical.

$$(4) X \langle V, W \rangle = X(f^2 \langle V, W \rangle_F) = 2f \langle V, W \rangle_F X(f). \quad \square$$

The following proposition generalizes the properties of the Levi-Civita connection for the warped product of (non-degenerate) semi-Riemannian manifolds (*cf. e.g.* [9], p. 206), to the degenerate case. We preferred to express them in terms of the Koszul form, and to give the proof explicitly, because for degenerate metric the Levi-Civita connection is not defined, and we need to avoid the index raising.

Proposition 2.5. Let $B \times_f F$ be a warped product, and let be the vector fields $X, Y, Z \in \mathfrak{L}(B \times F, B)$ and $U, V, W \in \mathfrak{L}(B \times F, F)$. Let \mathcal{K} be the Koszul form on $B \times_f F$, and $\mathcal{K}_B, \mathcal{K}_F$ the lifts of the Koszul forms on B , respectively F . Then

- (1) $\mathcal{K}(X, Y, Z) = \mathcal{K}_B(X, Y, Z)$.
- (2) $\mathcal{K}(X, Y, W) = \mathcal{K}(X, W, Y) = \mathcal{K}(W, X, Y) = 0$.
- (3) $\mathcal{K}(X, V, W) = \mathcal{K}(V, X, W) = -\mathcal{K}(V, W, X) = f\langle V, W \rangle_F X(f)$.
- (4) $\mathcal{K}(U, V, W) = f^2 \mathcal{K}_F(U, V, W)$.

Proof. (1) and (4) follow from properties of the lifts of vector fields, the Definition 1.8 of the Koszul form, and the equation (20).

(2) By Definition 1.8,

$$\begin{aligned} \mathcal{K}(X, Y, W) &= \frac{1}{2} \{ X\langle Y, W \rangle + Y\langle W, X \rangle - W\langle X, Y \rangle \\ &\quad - \langle X, [Y, W] \rangle + \langle Y, [W, X] \rangle + \langle W, [X, Y] \rangle \} \end{aligned}$$

We apply the Proposition 2.4. From the relation (1), $\langle Y, W \rangle = \langle W, X \rangle = \langle W, [X, Y] \rangle = 0$, from the relation (2) $[Y, W] = [W, X] = 0$, from the relation (3) $W\langle X, Y \rangle = 0$. Therefore $\mathcal{K}(X, Y, W) = 0$.

From (5) of the Theorem 1.9 we obtain that

$$\mathcal{K}(X, W, Y) = X\langle W, Y \rangle - \mathcal{K}(X, Y, W) = 0.$$

From (6) of the Theorem 1.9 and from Proposition 2.4(2) we obtain that

$$\mathcal{K}(W, X, Y) = \mathcal{K}(X, W, Y) - \langle [X, W], Y \rangle = 0.$$

$$\begin{aligned} \mathcal{K}(X, V, W) &:= \frac{1}{2} \{ X\langle V, W \rangle + V\langle W, X \rangle - W\langle X, V \rangle \\ &\quad - \langle X, [V, W] \rangle + \langle V, [W, X] \rangle + \langle W, [X, V] \rangle \} \\ (3) \quad &= \frac{1}{2} X\langle V, W \rangle \end{aligned}$$

from Proposition 2.4, using it as in the property (2) of the present Proposition. By applying the property (4) we have $\mathcal{K}(X, V, W) = f\langle V, W \rangle_F X(f)$. From Theorem 1.9 property (6),

$$\mathcal{K}(V, X, W) = \mathcal{K}(X, V, W) - \langle [X, V], W \rangle,$$

but since $[X, V] = 0$, $\mathcal{K}(V, X, W) = f\langle V, W \rangle_F X(f)$ as well.

From Theorem 1.9 property (5),

$$\mathcal{K}(V, W, X) = V\langle W, X \rangle - \mathcal{K}(V, X, W),$$

but since $\langle W, X \rangle = 0$, the property (3) of the present Proposition shows that

$$\mathcal{K}(V, W, X) = -f\langle V, W \rangle_F X(f).$$

□

Further, we will study some properties of the warped products, in situations when the warping function f is allowed to cancel or to become negative, and when (B, g_B) and (F, g_F) are allowed to be singular and with variable signature. But first, we need to recall what we know about non-degenerate warped products of non-singular semi-Riemannian manifolds.

3. Non-degenerate warped products

Here we recall for comparison and without proofs some fundamental properties of non-degenerate warped products between non-singular semi-Riemannian manifolds. The main reference is [9], p. 204–211. Here, (B, g_B) and (F, g_F) are semi-Riemannian manifolds, $f \in \mathcal{F}(B)$ a smooth function so that $f > 0$, and $B \times_f F$ the warped product of B and F .

For the proofs of the next propositions, see for example [9].

Proposition 3.1 (*cf.* [9], p. 206–207). Let $B \times_f F$ be a warped product, and let be the vector fields $X, Y \in \mathfrak{L}(B \times F, B)$ and $V, W \in \mathfrak{L}(B \times F, F)$. Let $\nabla, \nabla^B, \nabla^F$ be the Levi-Civita connections on $B \times_f F, B$, respectively F . Then

- (1) $\nabla_X Y$ is the lift of $\nabla_X^B Y$.
- (2) $\nabla_X V = \nabla_V X = \frac{Xf}{f} V$.
- (3) $\nabla_V W = -\frac{\langle V, W \rangle}{f} \text{grad } f + \widetilde{\nabla_V^F W}$, where $\widetilde{\nabla_V^F W}$ is the lift of $\nabla_V^F W$.

□

Proposition 3.2 (*cf.* [9], p. 209–210). Let $B \times_f F$ be a warped product, and R_B, R_F the lifts of the Riemann curvature tensors on B and F . Let be the vector fields $X, Y, Z \in \mathfrak{L}(B \times F, B)$ and $U, V, W \in \mathfrak{L}(B \times F, F)$, and let H^f be the *Hessian* of f , $H^f(X, Y) = \langle \nabla_X(\text{grad } f), Y \rangle_B$. Then

- (1) $R(X, Y)Z \in \mathfrak{L}(B \times F, B)$ is the lift of $R_B(X, Y)Z$.
- (2) $R(V, X)Y = -\frac{H^f(X, Y)}{f} V$.
- (3) $R(X, Y)V = R(V, W)X = 0$.
- (4) $R(X, V)W = -\frac{\langle V, W \rangle}{f} \nabla_X(\text{grad } f)$.

$$(5) \quad R(V, W)U = R_F(V, W)U + \frac{\langle \text{grad } f, \text{grad } f \rangle}{f^2} (\langle V, U \rangle W - \langle W, U \rangle V).$$

□

Corollary 3.3 (cf. [9], p. 211). Let $B \times_f F$ be a warped product, with $\dim F > 1$, and let be the vector fields $X, Y \in \mathfrak{L}(B \times F, B)$ and $V, W \in \mathfrak{L}(B \times F, F)$. Then

$$\begin{aligned} (1) \quad & \text{Ric}(X, Y) = \text{Ric}_B(X, Y) + \frac{\dim F}{f} H^f(X, Y). \\ (2) \quad & \text{Ric}(X, V) = 0. \\ (3) \quad & \text{Ric}(V, W) = \text{Ric}_F(V, W) \\ & + \left(\frac{\Delta f}{f} + (\dim F - 1) \frac{\langle \text{grad } f, \text{grad } f \rangle}{f^2} \right) \langle V, W \rangle. \end{aligned}$$

□

Corollary 3.4 (cf. [9], p. 211). Let $B \times_f F$ be a warped product, with $\dim F > 1$. Then, the scalar curvature s of $B \times_f F$ is related to the scalar curvatures s_B and s_F of B and F by

$$(21) \quad s = s_B + \frac{s_F}{f^2} + 2 \dim F \frac{\Delta f}{f} + \dim F (\dim F - 1) \frac{\langle \text{grad } f, \text{grad } f \rangle_B}{f^2}.$$

□

4. Warped products of semi-regular semi-Riemannian manifolds

In the following we will provide the condition for a degenerate warped product of semi-regular semi-Riemannian manifolds to be a semi-regular semi-Riemannian manifold.

Theorem 4.1. Let (B, g_B) and (F, g_F) be two radical-stationary semi-Riemannian manifolds, and $f \in \mathcal{F}(B)$ a smooth function so that $df \in \mathcal{A}^\bullet(B)$. Then, the warped product manifold $B \times_f F$ is a radical-stationary semi-Riemannian manifold.

Proof. We have to show that $\mathcal{K}(X, Y, W) = 0$ for any $X, Y \in \mathfrak{X}(B \times_f F)$ and $W \in \mathfrak{X}_\circ(B \times_f F)$. It is enough to check this for vector fields which are lifts of vector fields $X_B, Y_B, W_B \in \mathfrak{L}(B \times F, B)$, $X_F, Y_F, W_F \in \mathfrak{L}(B \times F, F)$, where $W_B, W_F \in \mathfrak{X}_\circ(B \times_f F)$. Then, from the Proposition 2.5:

$$(1) \quad \mathcal{K}(X_B, Y_B, W_B) = \mathcal{K}_B(X_B, Y_B, W_B) = 0,$$

- (2) $\mathcal{K}(X_B, Y_B, W_F) = \mathcal{K}(X_B, Y_F, W_B) = \mathcal{K}(X_F, Y_B, W_B) = 0$,
- (3) $\mathcal{K}(X_B, Y_F, W_F) = \mathcal{K}(Y_F, X_B, W_F) = f\langle Y_F, W_F \rangle_F X_B(f) = 0$,
because $\langle Y_F, W_F \rangle_F = 0$, and
 $\mathcal{K}(X_F, Y_F, W_B) = -f\langle X_F, Y_F \rangle_F W_B(f) = 0$, from $W_B(f) = 0$,
- (4) $\mathcal{K}(X_F, Y_F, W_F) = f^2 \mathcal{K}_F(X_F, Y_F, W_F) = 0$.

□

Theorem 4.2. Let (B, g_B) and (F, g_F) be two semi-regular semi-Riemannian manifolds, and $f \in \mathcal{F}(B)$ a smooth function so that $df \in \mathcal{A}^{\bullet 1}(B)$. Then, the warped product manifold $B \times_f F$ is a semi-regular semi-Riemannian manifold.

Proof. All contractions of the form $\mathcal{K}(X, Y, \bullet)\mathcal{K}(Z, T, \bullet)$ are well defined, according to Theorem 4.1. From Proposition 1.16, it is enough to show that they are smooth. It is enough to check this for vector fields which are lifts of vector fields $X_B, Y_B, Z_B, T_B \in \mathfrak{L}(B \times F, B)$, $X_F, Y_F, Z_F, T_F \in \mathfrak{L}(B \times F, F)$. Let's denote by \bullet_B and \bullet_F the symbol for the covariant contraction on B , respectively F . Then, from the Proposition 2.5:

$$\begin{aligned}
\mathcal{K}(X_B, Y_B, \bullet)\mathcal{K}(Z_B, T_B, \bullet) &= \mathcal{K}(X_B, Y_B, \bullet_B)\mathcal{K}(Z_B, T_B, \bullet_B) \\
&\quad + \mathcal{K}(X_B, Y_B, \bullet_F)\mathcal{K}(Z_B, T_B, \bullet_F) \\
&= \mathcal{K}_B(X_B, Y_B, \bullet_B)\mathcal{K}_B(Z_B, T_B, \bullet_B) \\
&\in \mathcal{F}(B \times_f F). \\
\mathcal{K}(X_B, Y_B, \bullet)\mathcal{K}(Z_F, T_B, \bullet) &= \mathcal{K}(X_B, Y_B, \bullet)\mathcal{K}(Z_B, T_F, \bullet) \\
&= \mathcal{K}(X_B, Y_B, \bullet_B)\mathcal{K}(Z_B, T_F, \bullet_B) \\
&\quad + \mathcal{K}(X_B, Y_B, \bullet_F)\mathcal{K}(Z_B, T_F, \bullet_F) = 0. \\
\mathcal{K}(X_B, Y_B, \bullet)\mathcal{K}(Z_F, T_F, \bullet) &= \mathcal{K}(X_B, Y_B, \bullet_B)\mathcal{K}(Z_F, T_F, \bullet_B) \\
&\quad + \mathcal{K}(X_B, Y_B, \bullet_F)\mathcal{K}(Z_F, T_F, \bullet_F) \\
&= -\mathcal{K}_B(X_B, Y_B, \bullet_B)f\langle Z_F, T_F \rangle_F df(\bullet_B) \\
&= -f\langle Z_F, T_F \rangle_F (\nabla_{X_B}^B Y_B)(df) \\
&\in \mathcal{F}(B \times_f F). \\
\mathcal{K}(X_B, Y_F, \bullet)\mathcal{K}(T_F, Z_B, \bullet) &= \mathcal{K}(X_B, Y_F, \bullet)\mathcal{K}(Z_B, T_F, \bullet) \\
&= \mathcal{K}(X_B, Y_F, \bullet_B)\mathcal{K}(Z_B, T_F, \bullet_B) \\
&\quad + \mathcal{K}(X_B, Y_F, \bullet_F)\mathcal{K}(Z_B, T_F, \bullet_F) \\
&= f\langle Y_F, \bullet_F \rangle_F X_B(f)\mathcal{K}(Z_B, T_F, \bullet_F) \\
&= f^3 X_B(f)\mathcal{K}_F(Z_B, T_F, Y_F) \\
&\in \mathcal{F}(B \times_f F). \\
\mathcal{K}(X_B, Y_F, \bullet)\mathcal{K}(Z_F, T_F, \bullet) &= \mathcal{K}(X_B, Y_F, \bullet_B)\mathcal{K}(Z_F, T_F, \bullet_B) \\
&\quad + \mathcal{K}(X_B, Y_F, \bullet_F)\mathcal{K}(Z_F, T_F, \bullet_F) \\
&= f^3 X_B(f)\langle Y_F, \bullet_F \rangle_F \mathcal{K}_F(Z_F, T_F, \bullet_F) \\
&= f^3 X_B(f)\mathcal{K}_F(Z_F, T_F, Y_F) \\
&\in \mathcal{F}(B \times_f F).
\end{aligned}$$

□

Remark 4.3. Even though (B, g_B) and (F, g_F) are non-degenerate semi-Riemannian manifolds, if the function f becomes 0, the warped product manifold $B \times_f F$ is a singular semi-Riemannian manifold.

Corollary 4.4. Let's consider that (B, g_B) is a non-degenerate semi-Riemannian manifold, and let $f \in \mathcal{F}(B)$. If (F, g_F) is radical-stationary, then the warped product $B \times_f F$ also is radical-stationary. If (F, g_F) is semi-regular, then the warped product $B \times_f F$ also is semi-regular. In particular, if both manifolds (B, g_B) and (F, g_F) are non-degenerate, and the warping function $f \in \mathcal{F}(B)$, then $B \times_f F$ is semi-regular.

Proof. If the manifold (B, g_B) is non-degenerate, then any function $f \in \mathcal{F}(B)$ also satisfies $df \in \mathcal{A}^\bullet(B)$ and $df \in \mathcal{A}^{\bullet 1}(B)$. Then the corollary follows from Theorems 4.1 and 4.2. \square

Proposition 4.5 (The case $f \equiv 0$). $B \times_0 F$ is a singular semi-Riemannian manifold with degenerate metric of constant rank $g = \dim B$.

Proof. The proof can be found in [7], p. 287. In fact, Kupeli does even more in [7], by showing that any radical-stationary semi-Riemannian manifold is locally a warped product of the form $B \times_0 F$. \square

Remark 4.6. The warped product of non-degenerate semi-Riemannian manifolds stays non-degenerate for $f > 0$. If $f \rightarrow 0$, we can see for example from [9] that the connection ∇ ([9], p. 206–207), the Riemann curvature R_∇ ([9], p. 209–210), the Ricci tensor Ric and the scalar curvature s ([9], p. 211) diverge in general.

5. Riemann curvature of semi-regular warped products

In this section we will assume (B, g_B) and (F, g_F) to be semi-regular semi-Riemannian manifolds, $f \in \mathcal{F}(B)$ a smooth function so that $df \in \mathcal{A}^{\bullet 1}(B)$, and $B \times_f F$ the warped product of B and F . The central point is to find the relation between the Riemann curvature R of $B \times_f F$ and those on (B, g_B) and (F, g_F) . The relations are similar to those for the non-degenerate case (cf. [9], p. 210–211) for the Riemann curvature operator $R(-, -)$, but since this operator is not well defined and is divergent for degenerate metric, we need to use the Riemann curvature tensor $R(-, -, -, -)$. The proofs given here are based only on formulae which work for the degenerate case as well.

Definition 5.1. Let (M, g) be a semi-regular semi-Riemannian manifold. The *Hessian* of a scalar field f satisfying $df \in \mathcal{A}^{\bullet 1}(M)$ is the

smooth tensor field $H^f \in \mathcal{T}_2^0 M$ defined by

$$(22) \quad H^f(X, Y) := (\nabla_X \mathrm{d}f)(Y)$$

for any $X, Y \in \mathfrak{X}(M)$.

Theorem 5.2. Let $B \times_f F$ be a degenerate warped product of semi-regular semi-Riemannian manifolds with $f \in \mathcal{F}(B)$ a smooth function so that $\mathrm{d}f \in \mathcal{A}^{\bullet 1}(B)$, and R_B, R_F the lifts of the Riemann curvature tensors on B and F . Let $X, Y, Z, T \in \mathfrak{L}(B \times F, B)$, $U, V, W, Q \in \mathfrak{L}(B \times F, F)$, and let H^f be the *Hessian* of f (which exists because $\mathrm{d}f \in \mathcal{A}^{\bullet 1}(B)$, see Definition 5.1. Then:

- (1) $R(X, Y, Z, T) = R_B(X, Y, Z, T)$
- (2) $R(X, Y, Z, Q) = 0$
- (3) $R(X, Y, W, Q) = 0$
- (4) $R(U, V, Z, Q) = 0$
- (5) $R(X, V, W, T) = -f H^f(X, T) \langle V, W \rangle_F$
- (6) $R(U, V, W, Q) = R_F(U, V, W, Q)$
 $+ f^2 \langle \langle \mathrm{d}f, \mathrm{d}f \rangle \rangle_{\bullet B} (\langle U, W \rangle_F \langle V, Q \rangle_F$
 $- \langle V, W \rangle_F \langle U, Q \rangle_F)$

the other cases being obtained by the symmetries of the Riemann curvature tensor.

Proof. In order to prove these identities, we will use the Koszul formula for the Riemann curvature from equation (18). We will denote the covariant contraction with \bullet on $B \times_f F$, and with $\overset{B}{\bullet}$ and $\overset{F}{\bullet}$ on B , respectively F .

$$\begin{aligned} (1) \quad R(X, Y, Z, T) &= X\mathcal{K}(Y, Z, T) - Y\mathcal{K}(X, Z, T) - \mathcal{K}([X, Y], Z, T) \\ &\quad + \mathcal{K}(X, Z, \bullet)\mathcal{K}(Y, T, \bullet) - \mathcal{K}(Y, Z, \bullet)\mathcal{K}(X, T, \bullet) \\ &= X\mathcal{K}(Y, Z, T) - Y\mathcal{K}(X, Z, T) - \mathcal{K}([X, Y], Z, T) \\ &\quad + \mathcal{K}(X, Z, \overset{B}{\bullet})\mathcal{K}(Y, T, \overset{B}{\bullet}) - \mathcal{K}(Y, Z, \overset{B}{\bullet})\mathcal{K}(X, T, \overset{B}{\bullet}) \\ &= R_B(X, Y, Z, T) \end{aligned}$$

where we applied (2) from the Proposition 2.5.

$$\begin{aligned} (2) \quad R(X, Y, Z, Q) &= X\mathcal{K}(Y, Z, Q) - Y\mathcal{K}(X, Z, Q) - \mathcal{K}([X, Y], Z, Q) \\ &\quad + \mathcal{K}(X, Z, \bullet)\mathcal{K}(Y, Q, \bullet) - \mathcal{K}(Y, Z, \bullet)\mathcal{K}(X, Q, \bullet) \\ &= \mathcal{K}(X, Z, \bullet)\mathcal{K}(Y, Q, \bullet) - \mathcal{K}(Y, Z, \bullet)\mathcal{K}(X, Q, \bullet) \\ &= \mathcal{K}(X, Z, \overset{B}{\bullet})\mathcal{K}(Y, Q, \overset{B}{\bullet}) - \mathcal{K}(Y, Z, \overset{B}{\bullet})\mathcal{K}(X, Q, \overset{B}{\bullet}) \\ &= 0 \end{aligned}$$

by the same property, which also leads to

$$\begin{aligned}
 (3) \quad R(X, Y, W, Q) &= X\mathcal{K}(Y, W, Q) - Y\mathcal{K}(X, W, Q) - \mathcal{K}([X, Y], W, Q) \\
 &\quad + \mathcal{K}(X, W, \bullet)\mathcal{K}(Y, Q, \bullet) - \mathcal{K}(Y, W, \bullet)\mathcal{K}(X, Q, \bullet) \\
 &= \mathcal{K}(X, W, \bullet)\mathcal{K}(Y, Q, \bullet) - \mathcal{K}(Y, W, \bullet)\mathcal{K}(X, Q, \bullet) \\
 &= 0.
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad R(U, V, Z, Q) &= U\mathcal{K}(V, Z, Q) - V\mathcal{K}(U, Z, Q) - \mathcal{K}([U, V], Z, Q) \\
 &\quad + \mathcal{K}(U, Z, \bullet)\mathcal{K}(V, Q, \bullet) - \mathcal{K}(V, Z, \bullet)\mathcal{K}(U, Q, \bullet) \\
 &= U(f\langle V, Q \rangle_F Z(f)) - V(f\langle U, Q \rangle_F Z(f)) \\
 &\quad - f\langle [U, V], Q \rangle_F Z(f) \\
 &\quad + \mathcal{K}(U, Z, \frac{B}{\bullet})\mathcal{K}(V, Q, \frac{B}{\bullet}) - \mathcal{K}(V, Z, \frac{B}{\bullet})\mathcal{K}(U, Q, \frac{B}{\bullet}) \\
 &\quad + \mathcal{K}(U, Z, \frac{F}{\bullet})\mathcal{K}(V, Q, \frac{F}{\bullet}) - \mathcal{K}(V, Z, \frac{F}{\bullet})\mathcal{K}(U, Q, \frac{F}{\bullet}) \\
 &= fZ(f)(U\langle V, Q \rangle_F - V\langle U, Q \rangle_F - \langle [U, V], Q \rangle_F) \\
 &\quad + \mathcal{K}(U, Z, \frac{F}{\bullet})\mathcal{K}(V, Q, \frac{F}{\bullet})_F - \mathcal{K}(V, Z, \frac{F}{\bullet})\mathcal{K}(U, Q, \frac{F}{\bullet})_F \\
 &= fZ(f)(U\langle V, Q \rangle_F - V\langle U, Q \rangle_F - \langle [U, V], Q \rangle_F) \\
 &\quad + f\langle U, \frac{F}{\bullet} \rangle_F Z(f)\mathcal{K}(V, Q, \frac{F}{\bullet})_F \\
 &\quad - f\langle V, \frac{F}{\bullet} \rangle_F Z(f)\mathcal{K}(U, Q, \frac{F}{\bullet})_F \\
 &= fZ(f)(U\langle V, Q \rangle_F - V\langle U, Q \rangle_F - \langle [U, V], Q \rangle_F) \\
 &\quad + \mathcal{K}(V, Q, U)_F - \mathcal{K}(U, Q, V)_F \\
 &= 0
 \end{aligned}$$

where we used (3) and (4) from the Proposition 2.5, together with the Definition 1.8. We also used the property that the covariant contraction on F cancels the coefficient f^2 of $\mathcal{K}(U, V, W)_F$.

$$\begin{aligned}
 (5) \quad R(X, V, W, T) &= X\mathcal{K}(V, W, T) - V\mathcal{K}(X, W, T) - \mathcal{K}([X, V], W, T) \\
 &\quad + \mathcal{K}(X, W, \bullet)\mathcal{K}(V, T, \bullet) - \mathcal{K}(V, W, \bullet)\mathcal{K}(X, T, \bullet) \\
 &= -X(fT(f)\langle V, W \rangle_F) \\
 &\quad - \mathcal{K}(V, W, \frac{B}{\bullet})\mathcal{K}(X, T, \frac{B}{\bullet}) \\
 &\quad + \mathcal{K}(X, W, \frac{F}{\bullet})\mathcal{K}(V, T, \frac{F}{\bullet})_F \\
 &= -X(fT(f)\langle V, W \rangle_F) \\
 &\quad + f\langle V, W \rangle_F df(\bullet)\mathcal{K}(X, T, \frac{B}{\bullet})_B \\
 &\quad + X(f)\langle W, \frac{F}{\bullet} \rangle_F T(f)\langle V, \frac{F}{\bullet} \rangle_F \\
 &= -X(f)T(f)\langle V, W \rangle_F - fX(T(f))\langle V, W \rangle_F \\
 &\quad + f\langle V, W \rangle_F \mathcal{K}(X, T, \frac{B}{\bullet})_B df(\frac{B}{\bullet}) \\
 &\quad + X(f)T(f)\langle W, V \rangle_F \\
 &= f\langle V, W \rangle_F \left[\mathcal{K}(X, T, \frac{B}{\bullet})_B df(\frac{B}{\bullet}) - X(T(f)) \right] \\
 &= f\langle V, W \rangle_F \left[\mathcal{K}(X, T, \frac{B}{\bullet})_B df(\frac{B}{\bullet}) - X\langle T, \text{grad } f \rangle_B \right] \\
 &= -fH^f(X, T)\langle V, W \rangle_F
 \end{aligned}$$

where we applied the definition of the Hessian for semi-regular semi-Riemannian manifolds, for f so that $df \in \mathcal{A}^{\bullet 1}(B)$, and the properties of the Koszul derivative of warped products, as in the Proposition 2.5.

$$\begin{aligned}
(6) \quad R(U, V, W, Q) &= UK(V, W, Q) - VK(U, W, Q) - \mathcal{K}([U, V], W, Q) \\
&\quad + \mathcal{K}(U, W, \bullet)\mathcal{K}(V, Q, \bullet) - \mathcal{K}(V, W, \bullet)\mathcal{K}(U, Q, \bullet) \\
&= R_F(U, V, W, Q) \\
&\quad + \mathcal{K}(U, W, \frac{B}{\bullet})\mathcal{K}(V, Q, \frac{B}{\bullet}) - \mathcal{K}(V, W, \frac{B}{\bullet})\mathcal{K}(U, Q, \frac{B}{\bullet}) \\
&= R_F(U, V, W, Q) \\
&\quad + f^2 \langle U, W \rangle_F df(\frac{B}{\bullet}) \langle V, Q \rangle_F df(\frac{B}{\bullet}) \\
&\quad - f^2 \langle V, W \rangle_F df(\frac{B}{\bullet}) \langle U, Q \rangle_F df(\frac{B}{\bullet}) \\
&= R_F(U, V, W, Q) \\
&\quad + f^2 \langle df, df \rangle_{\bullet B} (\langle U, W \rangle_F \langle V, Q \rangle_F \\
&\quad - \langle V, W \rangle_F \langle U, Q \rangle_F)
\end{aligned}$$

□

Remark 5.3. Despite the fact that the Riemann tensor $R(-, -)$ is divergent when the warping function converges to 0 even for warped products of non-degenerate metrics ([9], p. 209–210), Theorem 5.2 shows again that the Riemann curvature tensor $R(-, -, -, -)$ is smooth.

6. Polar and spherical warped products

In the following, we use the degenerate inner product of semi-regular manifolds to construct other manifolds. We start by providing a recipe to obtain from warped products spherical solutions of various dimension.

6.1. Polar warped products

Let $\mu, \rho \in \mathcal{F}(\mathbb{R})$ be smooth real functions so that $\mu^2(-r) = \mu^2(r)$ and $\rho^2(-r) = \rho^2(r)$ for any $r \in \mathbb{R}$, $i \in \{1, 2\}$. We can construct the following warped products between the spaces $(\mathbb{R}, \pm\mu^2 dr \otimes dr)$ and S^1 :

$$(23) \quad (\mathbb{R} \times_r S^1, \pm\mu^2 dr \otimes dr + \rho^2 d\vartheta \otimes d\vartheta).$$

We define on $\mathbb{R} \times_r S^1$ the equivalence relation $(r_1, \vartheta_1) \sim (r_2, \vartheta_2)$ if and only if either $r_1 = r_2$ and $\vartheta_1 = \vartheta_2$, or $r_1 = -r_2$ and $\vartheta_1 \equiv (\vartheta_2 + \pi) \bmod 2\pi$.

Definition 6.1. The manifold $(M, g) := (\mathbb{R}, \pm\mu^2 dr \otimes dr) \times_\rho S^1 / \sim$ is named the *polar warped product* between $(\mathbb{R}, \pm\mu^2 dr \otimes dr)$ and S^1 .

The manifold M is diffeomorphic to \mathbb{R}^2 .

We are looking for conditions which ensure the smoothness of the metric g on M .

Proposition 6.2. The metric g on M is smooth if and only if the following limit exists and is smooth:

$$(24) \quad \lim_{r \rightarrow 0} \frac{\pm \mu^2 r^2 - \rho^2}{r^4}.$$

Proof. The metric on $\mathbb{R}^2 - \{(0, 0)\}$ is, in Cartesian coordinates:

$$(25) \quad \begin{aligned} g &= \frac{1}{r^2} \begin{pmatrix} r \cos \vartheta & -\sin \vartheta \\ r \sin \vartheta & \cos \vartheta \end{pmatrix} \begin{pmatrix} \pm \mu^2 & 0 \\ 0 & \rho^2 \end{pmatrix} \begin{pmatrix} r \cos \vartheta & r \sin \vartheta \\ -\sin \vartheta & \cos \vartheta \end{pmatrix} \\ &= \frac{1}{r^2} \begin{pmatrix} \pm \mu^2 r^2 \cos^2 \vartheta + \rho^2 \sin^2 \vartheta & (\pm \mu^2 r^2 - \rho^2) \sin \vartheta \cos \vartheta \\ (\pm \mu^2 r^2 - \rho^2) \sin \vartheta \cos \vartheta & \pm \mu^2 r^2 \sin^2 \vartheta + \rho^2 \cos^2 \vartheta \end{pmatrix} \\ &= \frac{1}{r^2} \begin{pmatrix} \pm \mu^2 r^2 - (\pm \mu^2 r^2 - \rho^2) \sin^2 \vartheta & (\pm \mu^2 r^2 - \rho^2) \sin \vartheta \cos \vartheta \\ (\pm \mu^2 r^2 - \rho^2) \sin \vartheta \cos \vartheta & \pm \mu^2 r^2 - (\pm \mu^2 r^2 - \rho^2) \cos^2 \vartheta \end{pmatrix} \\ &= \frac{1}{r^2} \begin{pmatrix} \pm \mu^2 r^2 - \frac{\pm \mu^2 r^2 - \rho^2}{r^2} y^2 & \frac{\pm \mu^2 r^2 - \rho^2}{r^2} xy \\ \frac{\pm \mu^2 r^2 - \rho^2}{r^2} xy & \pm \mu^2 r^2 - \frac{\pm \mu^2 r^2 - \rho^2}{r^2} x^2 \end{pmatrix} \end{aligned}$$

Hence, g is smooth if and only if the limit (24) exists and is smooth. \square

Remark 6.3. The smoothness of g on M is ensured by the condition that $\rho^2(r) = \pm \mu^2 r^2 + u(r)r^4$ for some smooth function $u : \mathbb{R} \rightarrow \mathbb{R}$.

The metric becomes, in Cartesian coordinates,

$$(26) \quad g = \begin{pmatrix} \pm \mu^2 + uy^2 & -uxy \\ -uxy & \pm \mu^2 + ux^2 \end{pmatrix}$$

The determinant of the metric is

$$(27) \quad \det g = \mu^4 \pm u\mu^2 r^2,$$

and it follows that the metric becomes degenerate if $\mu = 0$ or $\mu^2 = \pm ur^2$.

Remark 6.4. If we want the metric to be semi-regular, we need to make sure that the equation (16) is respected. Since the coefficients μ and ρ depend only on r , it suffices that $\text{supp}(\partial_r \mu) \subseteq \text{supp}(\mu)$ and that there exists a smooth function $f \in \mathcal{F}(\mathbb{R})$ so that $\text{supp}(f) \subseteq \text{supp}(\mu)$ and

$$(28) \quad \frac{\partial \rho^2(r)}{\partial r} = f(r)\mu(r).$$

The next example shows how we can obtain the Euclidean plane \mathbb{R}^2 from a degenerate warped product.

Example 6.5. The flat metric on $\mathbb{R}^2 - \{(0,0)\}$ can be expressed in polar coordinates (r, ϑ) as

$$(29) \quad g = dr \otimes dr + r^2 d\vartheta \otimes d\vartheta.$$

The manifold $\mathbb{R}^2 - \{(0,0)\}$ can be obtained as the non-degenerate warped product $\mathbb{R}^+ \times_r S^1$, where $\mathbb{R}^+ = (0, \infty)$, with the natural metric dr^2 , and S^1 is the unit circle parameterized by ϑ , with the metric $d\vartheta^2$. The metric of $\mathbb{R}^+ \times_r S^1$ becomes degenerate at the point $r = 0$. We can use the degenerate warped product $\mathbb{R} \times_r S^1$, where the metric has the same form as in equation (29), and obtain a cylinder whose metric becomes degenerate at the points $r = 0$. The coordinate r is allowed here to become 0 or negative. The polar warped product $M = \mathbb{R} \times_r S^1 / \sim$ is isometric to the Euclidean space \mathbb{R}^2 .

The following example shows how we can obtain the sphere S^2 from a degenerate warped product.

Example 6.6. Let's rename the coordinate r to φ , let's take instead of $\rho(r)$ the function $\sin \varphi$, and let's make the metric on \mathbb{R} to be $d\varphi \otimes d\varphi$ (hence $\mu^2(\varphi) = 1$). Since $\sin \varphi = \varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \dots$, it follows that $\sin \varphi = \varphi - \varphi^3 h(\varphi)$, where h is a smooth function. Hence, $\sin^2 \varphi = \varphi^2 + u(\varphi)\varphi^4$, where $u(\varphi) = -2h(\varphi) + \varphi^2 h^2(\varphi)$ is a smooth function, and the smoothness of the metric g at $(0,0)$ is ensured. Let us now use instead the equivalence relation from §6.1, the relation defined by $(\varphi_1, \vartheta_1) \sim (\varphi_2, \vartheta_2)$ if and only if either $\varphi_1 \equiv \varphi_2 \pmod{2\pi}$ and $\vartheta_1 = \vartheta_2$, or $\varphi_1 \equiv -\varphi_2 \pmod{2\pi}$ and $\vartheta_1 \equiv (\vartheta_2 + \pi) \pmod{2\pi}$. We obtain the sphere $S^2 \cong \mathbb{R} \times_{\sin \varphi} S^1 / \sim$, having the metric

$$(30) \quad g_{S^2} = d\varphi \otimes d\varphi + \sin^2 \varphi d\vartheta \otimes d\vartheta.$$

The usual *spherical coordinates* can be obtained by restraining the coordinates (ϑ, φ) to the domain $[0, 2\pi) \times [0, \pi]$.

6.2. Spherical warped products

In a similar manner as in §6.1, we can define *spherical warped products*. We will work on $\mathbb{R} \times_\rho S^2$, where the sphere S^2 has the metric and parameterization as in the Example 6.6. The equivalence relation is defined as $(r_1, \vartheta_1, \varphi_1) \sim (r_2, \vartheta_2, \varphi_2)$ if and only if either $r_1 = r_2$ and $\vartheta_1 = \vartheta_2$ and $\varphi_1 = \varphi_2$, or $r_1 = -r_2$ and $\vartheta_1 \equiv (\vartheta_2 + \pi) \pmod{2\pi}$ and $\varphi_1 = \varphi_2$. We start with real smooth functions $\mu, \rho \in \mathcal{F}(\mathbb{R})$ so that $\mu^2(-r) = \mu^2(r)$ and $\rho^2(-r) = \rho^2(r)$ for any $r \in \mathbb{R}$, $i \in \{1, 2\}$, exactly

as in the polar case. We can construct the following warped products, between the spaces $(\mathbb{R}, \pm\mu^2 dr \otimes dr)$ and S^2 :

$$(31) \quad (\mathbb{R} \times \rho S^2, \pm\mu^2 dr \otimes dr + \rho^2 (d\varphi \otimes d\varphi + \sin^2 \varphi d\vartheta \otimes d\vartheta)).$$

Let $(M, g) = (\mathbb{R}, \pm\mu^2 dr \otimes dr) \times_\rho S^2 / \sim$. The manifold is $M = \mathbb{R}^3$. From §6.1 it follows that for any plane of M containing the axis $\mathbb{R} \times (0, 0)$ the smoothness results from the condition $\rho^2(r) = \pm\mu^2 r^2 + u(r)r^4$ for some function $u : \mathbb{R} \rightarrow \mathbb{R}$. The smoothness of g in these planes ensures its smoothness on the entire M . Moreover, by similar considerations it follows that M is semi-regular from the same condition given by the equation (28).

The same method can be used to obtain *n-spherical warped products*, by factoring the warped product $\mathbb{R} \times_\rho S^n$.

Example 6.7. As a direct application we can obtain the Euclidean space \mathbb{R}^3 in spherical coordinates from the degenerate warped product $\mathbb{R} \times_r S^2$.

Example 6.8. Similar to the Example 6.6, we can define an equivalence \sim so that the 3-sphere S^3 can be obtained as the spherical warped product $S^3 \cong \mathbb{R} \times_{\sin \gamma} S^2 / \sim$, having the metric

$$(32) \quad g_{S^3} = d\gamma \otimes d\gamma + \sin^2 \gamma (d\varphi \otimes d\varphi + \sin^2 \varphi d\vartheta \otimes d\vartheta).$$

Example 6.9. If in equation (30) we replace $\sin^2 \gamma$ with $\sinh^2 \gamma$, and \sim with the equivalence relation defined at the beginning of this section, we obtain the hyperbolic 3-space H^3 of constant sectional curvature -1 .

7. Applications of semi-regular warped products

In this section we apply the spherical warped product to construct cosmological models and to model black holes having semi-regular singularities.

The degenerate warped product allows the warping function to become 0 at some points. Under the hypothesis of the Theorem 1.18 the Riemann curvature still remains well-defined and smooth. As we shown in [12], for a smooth Riemann curvature tensor of a four-dimensional semi-regular manifold we can write a *densitized version of Einstein's equation* which remains smooth, and which reduces to the standard version if the metric is non-degenerate:

$$(33) \quad G \det g + \Lambda g \det g = \kappa T \det g,$$

where $G = \text{Ric} - \frac{1}{2}sg$, T is the stress-energy tensor, $\kappa := \frac{8\pi\mathcal{G}}{c^4}$, \mathcal{G} is Newton's constant and c the speed of light.

The generalization of the warped product we propose here provides a powerful method to resolve singularities in cosmology. If we show that a singularity can be obtained as a degenerate warped product of semi-regular (in particular non-degenerate) manifolds, it follows that the densitized version of the Einstein equation is smooth at that singularity.

7.1. Cosmological models

In the following we propose a generalization of the *Friedmann-Lemaître-Robertson-Walker* spacetime, allowing singularities.

If (Σ, g_Σ) is a connected three-dimensional Riemannian manifold of constant sectional curvature $k \in \{-1, 0, 1\}$ (i.e. H^3 , \mathbb{R}^3 or S^3) and $a \in (t_1, t_2)$, $-\infty \leq t_1 < t_2 \leq \infty$, $a \geq 0$, then the warped product $(t_1, t_2) \times_a \Sigma$ is called a *Friedmann-Lemaître-Robertson-Walker* spacetime:

$$(34) \quad g = -dt \otimes dt + a^2(t)g_\Sigma$$

For more generality, we use instead of the constant metric on \mathbb{R} , a metric $-\mu^2 dt \otimes dt$, where $\mu \in \mathcal{F}(\mathbb{R})$, and we allow a to become 0 or negative. The warped product becomes then

$$(35) \quad g = -\mu^2 dt \otimes dt + a^2(t)g_\Sigma$$

The singularities are semi-regular if there exists a smooth function $f \in \mathcal{F}(\mathbb{R})$ so that $\text{supp}(f) \subseteq \text{supp}(\mu)$ and

$$(36) \quad \frac{\partial a^2(t)}{\partial t} = f(t)\mu(t).$$

Example 7.1. By allowing a to become 0, we can study cosmological models in which the evolution equation can pass through the singularities¹.

Example 7.2. Another possible application of the semi-regular warped products to the Friedmann-Lemaître-Robertson-Walker spacetime is to make the Big Bang singularity to be semi-regular. The model is obtained as an n -spherical warped product, as in the section §6.2. The initial singularity is smooth if a is smooth on \mathbb{R} , and $a^2(t) = -\mu^2 t^2 + u(t)t^4$ for some function u . It is semi-regular if the condition from equation (36) is satisfied at $t = 0$. The condition $a^2(t) = -\mu^2 t^2 + u(t)t^4$ shows that near $t = 0$, $a^2(t) < 0$, and the metric becomes negative definite. This model is resemblant to the Hartle-Hawking one [4], except that in the

¹In the meantime, in [16], we applied the technique presented here in detail and we proved that the Friedmann-Lemaître-Robertson-Walker spacetime can be extended before the Big Bang singularity, and the densitized version of the Einstein equation remains smooth.

former the spacelike directions change the signature, while in the latter, the time direction changes it. In order to keep the metric Lorentzian, we have to make sure that $a^2 \geq 0$. For this, the function μ has to be of the form $\mu = \tilde{\mu}t$ for some smooth function $\tilde{\mu}$, and since

$$(37) \quad a^2 = (u - \tilde{\mu}^2)t^4$$

there has to exist a smooth function $\tilde{a} \geq 0$ satisfying

$$(38) \quad \tilde{a}^2 = u - \tilde{\mu}^2.$$

and

$$(39) \quad a = \tilde{a}t^2.$$

7.2. Spherical black holes

The spherically symmetric solutions (M, g) in General Relativity are usually of the form $M = \mathbb{R} \times \mathbb{R}^+ \times S^2$, and the metric g has the form

$$(40) \quad g = -\alpha_t^2(t, r)dt \otimes dt + \alpha_r^2(t, r)dr \otimes dr + \rho^2(t, r)g_{S^2},$$

where $\alpha_t, \alpha_r, \rho \in \mathcal{F}(\mathbb{R} \times \mathbb{R}^+)$ and g_{S^2} is given by the equation (30). This spacetime is in fact the warped product

$$(\mathbb{R} \times \mathbb{R}^+) \times_{\rho(t, r)} S^2$$

between the semi-Riemannian manifold

$$(\mathbb{R} \times \mathbb{R}^+, -\alpha_t^2(t, r)dt \otimes dt + \alpha_r^2(t, r)dr \otimes dr)$$

and the Riemannian manifold S^2 , with warping function $\rho(t, r)$. This example is known for non-degenerate metric (*cf. e.g.* [11], p. 228), and it usually has singularities at $r = 0$.

Let \widetilde{M} be a manifold $\mathbb{R}^2 \times_{\rho} S^2$, where the functions $\alpha_t, \alpha_r, \rho \in \mathcal{F}(\mathbb{R}^2)$ are taken so that α_t^2, α_r^2 and ρ are symmetric in r . We can choose α_t, α_r and ρ so that, by identifying each point $(t, r, \vartheta, \varphi)$ with $(t, -r, (\vartheta + \pi) \bmod 2\pi, \varphi)$, we obtain a smooth singular semi-Riemannian manifold M . By appropriately choosing them, we can make the singularities at $r = 0$ of \widetilde{M} to be semi-regular, or we can even remove them, similar to the examples from §6.

If at some points α_t or α_r becomes 0, the metric on \mathbb{R}^2 is degenerate. From the condition (16) it follows that if for any $a, b \in \{t, r\}$, $c \in \{a, b\}$, there are some functions $f_{abc} \in \mathcal{F}(\mathbb{R}^2)$ satisfying $\text{supp}(f_{abc}) \subseteq \text{supp}(\alpha_c)$, so that

$$(41) \quad \partial_a \alpha_b^2 = f_{abc} \alpha_c,$$

then the manifold $(\mathbb{R}^2, -\alpha_t^2(t, r)dt \otimes dt + \alpha_r^2(t, r)dr \otimes dr)$ is semi-regular. Let us now find a condition ensuring that $d\rho \in \mathcal{A}^{\bullet 1}(\mathbb{R}^2)$ (and, by the Theorem 4.2, that the manifold M is semi-regular). From Definition 1.12, $d\rho \in \mathcal{A}^{\bullet 1}(\mathbb{R}^2)$ is equivalent to $d\rho \in \mathcal{A}^{\bullet 1}(\mathbb{R}^2)$ and the condition

$$(42) \quad \langle \nabla_{\partial_a}^b \partial_b, d\rho \rangle_{\bullet} \in \mathcal{F}(\mathbb{R}^2)$$

for any $a, b \in \{t, r\}$. From $(d\rho)(X) = X(\rho)$ it follows that the equation (42) is equivalent to the existence of some functions $h_{abc} \in \mathcal{F}(\mathbb{R}^2)$ satisfying $\text{supp}(h_{abc}) \subseteq \text{supp}(\alpha_c)$, so that

$$(43) \quad \mathcal{K}_{abc} \partial_c \rho = h_{abc} \alpha_c^2.$$

Remember that the components of the Koszul form in a coordinate chart reduces to Christoffel's symbols of the first kind, which, when the metric is diagonal, are of the form $\pm \frac{1}{2} \partial_a \alpha_b^2$. The condition (43) becomes equivalent to a condition of the form

$$(44) \quad \partial_a \alpha_b^2 \partial_c \rho = h_{abc} \alpha_c^2$$

for all $a, b \in \{t, r\}$, $c \in \{a, b\}$. If there is a function $F_c \in \mathcal{F}(\mathbb{R}^2)$ so that

$$(45) \quad \partial_c \rho = F_c \alpha_c,$$

from (41), we can take $h_{abc} = f_{abc} F_c$. This ensures that $d\rho \in \mathcal{A}^{\bullet 1}(\mathbb{R}^2)$, and it follows that M is semi-regular.

Remark 7.3. This discussion suggests a possibility to construct black holes having semi-regular singularities, and being therefore compatible with the densitized version of Einstein's equation. Such singularities don't block the time evolution. They are compatible with a unitary evolution, and Hawking's information loss paradox [5, 6] may be avoided. We will develop this viewpoint in the following articles ².

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²In the meantime, we applied the ideas introduced in this article to resolve the singularities of the Schwarzschild [13], Reissner-Nordström [14] and Kerr-Newman [15] black holes.

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